Hand in no. 2, 4, 5 and 6 by November 7.

Assignment 7

- 1. Let f be continuously differentiable on [a, b]. Show that it has a differentiable inverse if and only if its derivative is not equal to 0 at every point. This is 2060 stuff.
- 2. Consider the function

$$f(x) = \frac{1}{2}x + x^2 \sin \frac{1}{x}, \quad x \neq 0,$$

and set f(0) = 0. Show that f is differentiable at 0 with f'(0) = 1/2 but it has no local inverse at 0. Does it contradict the Inverse Function Theorem?

- 3. Study the map on \mathbb{R}^2 given $(x, y) \mapsto (x^2 y^2, 2xy)$. Show that it is local invertible everywhere except at the origin. Does its inverse exist globally?
- 4. Consider the mapping from \mathbb{R}^2 to itself given by $f(x,y) = x x^2$, g(x,y) = y + xy. Show that it has a local inverse at (0,0). And then write down the inverse map so that its domain can be described explicitly.
- 5. Let F be a continuously differentiable map from the open $U \subset \mathbb{R}^n$ to \mathbb{R}^n whose Jacobian determinant is non-vanishing everywhere. Prove that it maps every open set in U to an open set, that is, F is an open map. Does its inverse $F^{-1}: F(U) \to U$ always exist?
- 6. Consider the function

$$h(x,y) = (x - y^2)(x - 3y^2), \quad (x,y) \in \mathbb{R}^2$$

Show that the set $\{(x, y) : h(x, y) = 0\}$ cannot be expressed as a local graph of a C^1 -function over the x or y-axis near the origin. Explain why the Implicit Function Theorem is not applicable.

7. Consider a real polynomial $p(x, \mathbf{a}) = a_0 + a_1 x + \dots + a_n x^n$ as a function of x and the coefficients. A point x_0 is a simple root of p if $p(x_0, \mathbf{a}) = 0$ and $p'(x_0, \mathbf{a}) \neq 0$ where $\mathbf{a} = (a_0, a_1, \dots, a_n)$. Let x_0 be a simple of $p(\cdot, \mathbf{a}_0)$. Show that there is a smooth function φ defined in an open set in \mathbb{R}^{n+1} containing \mathbf{a}_0 such that $x = \varphi(\mathbf{a})$ is a simple root for $p(\cdot, \mathbf{a}) = 0$. What happens when the root is simple.